# Spectral decompositions for evolution operators of mixing dynamical systems

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**Abstract.** Spectral decompositions for the evolution operator on an energy shell in phase space are constructed for the free motion on compact 2D surfaces of constant negative curvature. Applications to quantum chaos and in particular to the recently proposed ballistic $\sigma$-model are briefly discussed.

## 1. Introduction

The dynamics of a Hamiltonian system can be described either in terms of the trajectories $x_0 \rightarrow x(t) = U_t x_0$ in the phase space, or by specifying the laws of evolution of a function $\varphi(x)$ on the phase space: $\varphi \rightarrow \hat{U}_t \varphi$. The evolution operator $\hat{U}_t$, which advances a function along the trajectories, is defined by

$$\hat{U}_t \varphi(x) = \varphi(U_t x) = \varphi(x(t)).$$

Quantum mechanics has a natural relation to the trajectory based approach through Feynman's path integral. In the semiclassical limit the path integral can be approximated by the saddle point contributions (which turn out to be the classical orbits) and leads to the Gutzwiller trace formula [1] for the Green function of the quantum mechanical Hamiltonian. This is a useful tool for studying such problems as the quantum energy level correlations but it requires knowledge of the long periodic orbits in order to obtain the correlations at small energy differences [2,3]. The situation is particularly bad in chaotic systems, where the periodic orbits proliferate exponentially with length. In practice some uncontrolled approximations about correlations of actions for different periodic orbits are made to get the analytical results.

In contrast, the flow based approach seems suitable for studying the behaviour of chaotic systems at long times when due to the decay of correlations the dynamics becomes trivial. To use this approach in quantum chaos two problems however need to be overcome. First, it is unclear how to relate the quantum mechanics to the evolution operator since there is no analogue of the Feynman path integral. Second, one needs to be able to calculate various properties of the evolution operator which are naturally formulated in terms of the spectral decomposition of $\hat{U}_t$ in decaying eigenmodes.

The first problem has recently been addressed in [4,5] where it has been conjectured that the (suitably averaged) correlation functions of the quantum energy levels and/or quantum eigenfunctions can be generated from an effective action of nonlinear $\sigma$-model type involving the Liouville operator $\hat{L} = \frac{\mathrm{d}}{\mathrm{d}t} \hat{U}_t$ (the so-called 'ballistic' $\sigma$-model). The inspiration for this approach comes from the well developed theory of weakly disordered metals, where the disorder averaged properties of an ensemble of macroscopically identical systems are

calculated using a similar $\sigma$-model with a diffusion operator instead of the Liouvillian one. (See [6, 7] for a detailed discussion of the diffusive case.)

Our paper is devoted to the second problem. In particular we compute a generalized spectral decomposition of the evolution operator

$$\hat{U}_t = \sum_\lambda e^{\lambda t} |\lambda\rangle\langle\tilde{\lambda}| \tag{1}$$

for a class of 'model' chaotic systems, namely the free motion on two-dimensional compact surfaces with constant negative curvature.

For mixing systems the only square integrable eigenfunctions of $\hat{U}_t$ are the constant functions (see appendix A). The eigenfunctions $|\lambda\rangle$ entering the spectral decomposition belong to a larger space $C^\infty(M)^*$ of distributions and can be obtained from the residues of the analytical continuation of matrix elements of the resolvent of $\hat{L}$ (see section 2). The procedure was suggested by Ruelle [8] and was successfully used to study the dynamics of some chaotic maps (see [9–12]). The eigenvalues $\lambda$ entering the spectral decomposition are sometimes called 'Ruelle resonances'. For our model dynamical system the resonances can be found using the Selberg trace formula to relate the classical and quantum zeta-functions [13]. This approach however does not provide the eigenmodes, which we compute using the representation theory of $SO(2, 1)$.

The rest of the paper is organized as follows. We start by outlining the Ruelle procedure for flows, concentrating on criteria for convergence of the resulting decomposition (section 2). We then introduce geodesic flows on constant negative-curvature surfaces (section 3), summarize the important facts from the representation theory of $SO(2, 1)$ and proceed to obtaining the spectral decomposition (section 4, equation (36)). This decomposition is used to refine approximations of the decay of correlations for geodesic flows (section 5.1) and relate the evolution of particle density on the configuration space to the Laplacian operator (section 5.2). To conclude we discuss the regularization of the Liouvillian operator entering the ballistic $\sigma$-model (section 6.2). Some technical details are relegated to the appendices.

## 2. Decompositions of evolution operators for general mixing systems

### 2.1. Spectral decompositions

Consider a Hamiltonian system on a phase space $T^*N$ with coordinates $x = (q, p)$ where $N$ is a smooth compact manifold parametrized by $q$ and $p$ is the momentum. When the Hamiltonian $H$ does not depend on time energy is conserved and the trajectories lie on surfaces $M$ of constant energy. We study the restriction of the flow $U_t$ to a constant-energy shell $M$ for some value of the energy. If $dH$ is non-zero on $M$ the Liouvillian measure $dp\,dq$ induces a measure $d\mu$ on $M$ according to $d\mu\,dH = dp\,dq$ [13]. By Liouville's theorem this measure is preserved by the flow $U_t$.

The evolution operator $\hat{U}_t$ advances a function $\varphi(x) : M \to \mathbb{C}$ along the flow

$$\hat{U}_t\varphi(x) = \varphi(U_t x) = e^{\hat{L}t}\varphi(x)$$

where for a Hamiltonian system the Liouville operator $\hat{L}$ is given by

$$\hat{L}\varphi = \left.\frac{d}{dt}\right|_{t=0} \hat{U}_t\varphi = \{H, \varphi\} \tag{2}$$

with $\{ , \}$ being the Poisson bracket.

The operator $\hat{U}_t$ is unitary with respect to the scalar product

$$\langle\xi(x)|\varphi(x)\rangle = \int_{x\in M} \xi(x)\overline{\varphi(x)}\,d\mu(x) \tag{3}$$

since $\mu$ is preserved by $U_t$. A natural space for $\hat{U}_t$ is the Hilbert space $L_2(M)$ of square integrable functions on $M$, while $\hat{L}$ preserves a smaller space of compactly supported infinitely differentiable functions $C^\infty(M)$.

We would like to find a spectral decomposition (1) for the operator $\hat{U}_t$ in terms of its eigenvalues $e^{\lambda t}$ and projectors $\langle \tilde{\lambda}|$ onto its eigenfunctions $|\lambda\rangle$. This would enable us to establish the evolution of a function $\varphi(x)$:

$$\hat{U}_t|\varphi\rangle = \sum_\lambda e^{\lambda t}|\lambda\rangle\langle\tilde{\lambda}|\varphi\rangle.$$

Since the operator $\hat{U}_t$ is unitary on $L_2(M)$ it can only have eigenvalues which lie on the unit circle. On the other hand, in a mixing chaotic system (definitions and properties of mixing systems are given in appendix A) all deviations from a constant value decay, implying the existence of modes corresponding to eigenvalues with modulus less than one. The resolution of this apparent paradox lies in observing that the $L_2(M)$ eigenfunctions of $\hat{U}_t$ do not necessarily form a basis in $L_2(M)$ or even in $C^\infty(M)$. In fact, for a mixing system the only square integrable eigenfunctions of $\hat{U}_t$ are the constant functions which have the eigenvalue 1 (appendix A). In order to find a spectral decomposition for $\hat{U}_t$ we need to extend the Hilbert space $L_2(M)$ to a rigged Hilbert space as described below. A similar approach was successfully employed for various maps (see e.g. [9, 10]).

### 2.2. Rigged Hilbert spaces

A Hilbert space $\mathbb{H}$ may be extended to the set of linear functionals on a suitable dense subspace $\mathbb{S}$. The resulting space $\mathbb{S}^*$ is called the rigging of $\mathbb{H}$ over $\mathbb{S}$. We denote by $f[\varphi]$ the value of the linear functional $f \in \mathbb{S}^*$ on the vector $\varphi \in \mathbb{S}$.

A vector $g \in \mathbb{H}$ may be naturally embedded in $\mathbb{S}^*$ as $g[\varphi] = \langle\varphi|g\rangle$, so we get a sequence of spaces $\mathbb{S} \subset \mathbb{H} \subset \mathbb{S}^*$. We make the above embedding explicit [9] and denote the functional $f \in \mathbb{S}^*$ by $|f\rangle$, its value on a vector $\varphi \in \mathbb{S}$ being $f[\varphi] = \langle\varphi|f\rangle$. We also introduce the notation $\langle f|\varphi\rangle$ for the antilinear functional $\overline{f[\varphi]}$.

We search for eigenfunctionals of the evolution operator $\hat{U}_t$ in the rigging $C^\infty(M)^*$ of $L_2(M)$ over $C^\infty(M)$. The evolution operator is extended to $C^\infty(M)^*$ by

$$\langle\varphi|\hat{U}_t f\rangle = \langle\hat{U}_{-t}\varphi|f\rangle \qquad \text{where} \quad |f\rangle \in C^\infty(M)^* \quad \text{and} \quad \varphi \in C^\infty(M).$$

Since $\hat{U}_t$ does not preserve any scalar product in $C^\infty(M)^*$ its eigenvalues need not lie on the unit circle.

We shall construct decompositions for the correlation function

$$\langle\xi|\hat{U}_t|\varphi\rangle = \int_M d\mu\, \xi(x)\hat{U}_t\varphi(x)$$

in a subset of eigenfunctionals $\{|f_\lambda\rangle\}$ from $C^\infty(M)^*$:

$$\langle\xi|\hat{U}_t|\varphi\rangle = \sum_\lambda e^{\lambda t}\langle\xi|f_\lambda\rangle\langle f_{-\bar{\lambda}}|\varphi\rangle \qquad \hat{U}_t|f_\eta\rangle = e^{\eta t}|f_\eta\rangle. \tag{4}$$

In general (4) only has asymptotic meaning (see section 2.3 for details), and will only converge when $t > 0$ and $\xi$ and $\varphi$ belong to a subspace $\mathbb{T}$ of $C^\infty(M)$. For the free motion on compact surfaces of constant negative curvature we find such a subspace which is dense in the set of infinitely differentiable functions. Note that the eigenfunction $|f_{-\bar{\lambda}}\rangle$ with the eigenvalue $e^{-\bar{\lambda}t}$ appears in (4) due to the unitarity condition $(\hat{U}_t)^+ = U_{-t}$, which leads to another expansion for the correlation function:

$$\langle\xi|\hat{U}_t|\varphi\rangle = \overline{\langle\varphi|\hat{U}_{-t}|\xi\rangle} = \sum_\lambda e^{-\bar{\lambda}t}\langle\xi|f_{-\bar{\lambda}}\rangle\langle f_\lambda|\varphi\rangle \tag{5}$$

converging when $t < 0$ and $\xi, \varphi \in \mathbb{T}$.

Decompositions (4, 5) appear naturally in connection with the correlation function $\langle \xi | (\hat{L} - z)^{-1} | \phi \rangle$ of the resolvent of $\hat{L}$.

### 2.3. Resolvent method for calculating the evolution operator decompositions

Choosing an integral representation for the resolvent of the Liouville operator converging when $\operatorname{Re} z > 0$

$$R_-(z) = -\int_0^\infty e^{-zT} \hat{U}_T \, dT = (\hat{L} - z)^{-1} \tag{6}$$

and assuming decomposition (4) we obtain for the correlation function of the resolvent

$$F_{\xi,\varphi}(z) \equiv \langle \xi | R_-(z) | \varphi \rangle = \sum_\lambda \frac{\langle \xi | f_\lambda \rangle \langle f_{-\bar\lambda} | \varphi \rangle}{\lambda - z} \tag{7}$$

where $\xi, \varphi \in \mathbb{T}$.

Conversely decomposition (4) can be constructed by analytically continuing $F$ to the left-hand half of the $z$-plane and analysing its singularities and residues. The positions of the poles dictate the rate of decay of the correlation function of the evolution operator [8]. The values of $\lambda$ in decomposition (4) are called Ruelle resonances.

Each term in decomposition (4) is well defined for $\xi, \varphi \in C^\infty(M)$ when

the position of the poles of $F_{\xi,\varphi}(z)$

does not depend on the choice of $\xi$ or $\varphi$. $\tag{8}$

A famous conjecture due to Ruelle [8] states that the poles of the resolvent for a mixing system do indeed satisfy this condition.

The sum in (4) converges if in addition the residues $\operatorname{Res}(\lambda, F_{\xi,\varphi})$ grow slowly enough when $|\lambda| \to \infty$:

$$\lim_{R \to \infty} \sum_{R < |\lambda| < \infty} e^{\lambda t} \operatorname{Res}(\lambda, F_{\xi,\varphi}) = 0. \tag{9}$$

The set $\mathbb{T}$ will comprise of functions in $C^\infty(M)$ which satisfy condition (9).

To prove the convergence we consider the integral of $F_{\xi,\varphi,t}(z) = e^{zt} \langle \xi | R_-(z) | \phi \rangle$ for $\xi$ and $\phi$ in $\mathbb{T}$ around the contour $|z| = R$ for a given large $R$. In the limit $R \to \infty$ this integral converges due to (9) to

$$\lim_{R \to \infty} \int_{|z|=R} F_{\xi,\varphi,t}(z) \, dz = 2\pi i \sum_\lambda \operatorname{Res}(\lambda, F_{\xi,\varphi,t}) = 2\pi i \sum_\lambda e^{\lambda t} \operatorname{Res}(\lambda, F_{\xi,\varphi}) \tag{10}$$

where the sum is over all the poles $\lambda$ other than $\infty$.

Since all the poles are in the left-hand half of the $z$-plane the contour can be deformed to go along the line $\operatorname{Re} z = a$ for a fixed $a > 0$ leading to

$$\begin{aligned}
\int_{\operatorname{Re} z = a} F_t(z) \, dz &= \int_{\operatorname{Re} z = a} e^{zt} \int_{T=0}^\infty e^{-zT} \langle \xi | \hat{U}_T | \varphi \rangle \, dT \, dz \\
&= \int_{T=0}^\infty \int_{y=-\infty}^\infty e^{(a+iy)(t-T)} \, dy \, \langle \xi | \hat{U}_T | \varphi \rangle \, dT \\
&= \int_{T=0}^\infty \delta(t - T) \langle \xi | \hat{U}_T | \varphi \rangle \, dT \\
&= \langle \xi | \hat{U}_t | \varphi \rangle \qquad \text{for} \quad t > 0.
\end{aligned} \tag{11}$$

From equation (10) we get the decomposition into residues

$$\langle\xi|\hat{U}_t|\varphi\rangle = 2\pi\mathrm{i}\sum_\lambda \mathrm{e}^{\lambda t}\mathrm{Res}\,(\lambda, F) \tag{12}$$

which converges absolutely for $t > 0$.

The individual terms in (12) exist for arbitrary $\xi, \varphi \in C^\infty(M)$ but the series converges only for $\xi, \varphi \in \mathbb{T}$. The residue $\mathrm{Res}(\lambda, F)$ is a linear functional of $\xi$ and an antilinear functional of $\varphi$. We define the operator $\hat{K}_\lambda : C^\infty(M) \to C^\infty(M)^*$ by

$$2\pi\mathrm{i}\mathrm{Res}(\lambda, F) = \langle\xi|\hat{K}_\lambda|\varphi\rangle \qquad \xi, \varphi \in C^\infty(M)$$

so that

$$\langle\xi|\hat{U}_t|\varphi\rangle = \sum_\lambda \mathrm{e}^{\lambda t}\langle\xi|\hat{K}_\lambda|\varphi\rangle \qquad \xi, \varphi \in \mathbb{T}$$

and $\hat{K}_\lambda|\varphi\rangle$ is an eigenfunctional of $\hat{U}_t$

$$\hat{U}_t\hat{K}_\lambda|\varphi\rangle = \mathrm{e}^{\lambda t}\hat{K}_\lambda|\varphi\rangle.$$

Let $\{|f_\lambda^k\rangle\}$ be a basis for the eigenspace corresponding to the eigenvalue $\mathrm{e}^{\lambda t}$, i.e. the image of $C^\infty(M)$ under $\hat{K}_\lambda$.

$$\hat{K}_\lambda|\varphi\rangle = \sum_k \langle c_k|\varphi\rangle|f_\lambda^k\rangle \qquad \text{with} \quad \hat{U}_t|f_\lambda^k\rangle = \mathrm{e}^{\lambda t}|f_\lambda^k\rangle.$$

Substituting this expression into (12) we obtain

$$\langle\xi|\hat{U}_t|\varphi\rangle = \sum_\lambda \langle\xi|\hat{K}_\lambda|\varphi\rangle$$
$$= \sum_\lambda \sum_k \langle\xi|f_\lambda^k\rangle\langle f_{-\bar\lambda}^k|\varphi\rangle \qquad \xi, \varphi \in \mathbb{T} \quad t > 0 \tag{13}$$

where by virtue of the unitarity of the evolution operator the coefficients $\langle c_k|\varphi\rangle$ are given by eigenfunctionals $|f_{-\bar\lambda}^k\rangle$ of $\hat{U}_t$ with eigenvalues $\mathrm{e}^{-\bar\lambda t}$.

$$\langle c_k|\varphi\rangle = \overline{\langle\varphi|f_{-\bar\lambda}^k\rangle}.$$

Equation (13) reduces to (4) in the non-degenerate case when the image of $\hat{K}_\lambda$ is one dimensional.

For arbitrary $\xi$ and $\phi$ in $C^\infty(M)$ the convergence of decomposition (4) is asymptotic:

$$\left|\langle\xi|U_t|\varphi\rangle - \sum_{\mathrm{Re}\,\lambda\geqslant -a} \mathrm{e}^{\lambda t}\langle f_{-\bar\lambda}|\varphi\rangle\langle\xi|f_\lambda\rangle\right| < C(a)\mathrm{e}^{-at} \qquad a > 0. \tag{14}$$

This inequality holds since the integral

$$\int_0^\infty \left(\langle\xi|U_t|\varphi\rangle - \sum_{\mathrm{Re}\,\lambda\geqslant -a} \mathrm{e}^{\lambda t}\langle f_{-\bar\lambda}|\varphi\rangle\langle\xi|f_\lambda\rangle\right)\mathrm{e}^{-zt}\,\mathrm{d}t$$

is analytic in the region $\mathrm{Re}\,z > -a$ of the $z$-plane.

A similar procedure starting with the representation $R_+(z)$ for the resolvent converging at $\mathrm{Re}\,z < 0$

$$R_+(z) = \int_{-\infty}^0 \mathrm{e}^{zt}\hat{U}_t\,\mathrm{d}t = (\hat{L} - z)^{-1} \tag{15}$$

gives the decomposition (5), which converges absolutely for $t < 0$ when the conditions (8), (9) are met.

*2.4. Evolution operator at long times*

We can use the decompositions (4) and (5) to study the approach to equilibrium at long times for mixing dynamical systems.

We show below that for $\xi$ and $\varphi$ in $\mathbb{T}$ the function $C(a)$ in (14) is independent of $a$. The behaviour at long future times may be approximated by retaining only the terms in (4) where $\lambda$ has a small negative real part.

$$
\begin{aligned}
\langle \xi | \hat{U}_t | \varphi \rangle &= \sum_{\mathrm{Re}\,\lambda \geqslant -a} \mathrm{e}^{\lambda t} \langle f_{-\bar{\lambda}} | \varphi \rangle \langle \xi | f_\lambda \rangle + \sum_{\mathrm{Re}\,\lambda < -a} \mathrm{e}^{\lambda t} \langle f_{-\bar{\lambda}} | \varphi \rangle \langle \xi | f_\lambda \rangle \\
&\approx \sum_{\mathrm{Re}\,\lambda \geqslant -a} \mathrm{e}^{\lambda t} \langle f_{-\bar{\lambda}} | \varphi \rangle \langle \xi | f_\lambda \rangle \\
&= \langle \varphi | 1 \rangle \langle 1 | \xi \rangle + \sum_{0 > \mathrm{Re}\,\lambda \geqslant -a} \mathrm{e}^{\lambda t} \langle f_{-\bar{\lambda}} | \varphi \rangle \langle \xi | f_\lambda \rangle
\end{aligned}
\tag{16}
$$

where we have separated out the the contribution from the constant $L_2(M)$ eigenfunction with $\lambda = 0$. For a mixing system the latter is the only non-decaying eigenfunction. By the absolute convergence of the sum (4) for $t > 0$ the discarded terms are bounded by

$$
\left| \sum_{\mathrm{Re}\,\lambda < -a} \mathrm{e}^{\lambda t} \langle f_{-\bar{\lambda}} | \varphi \rangle \langle \xi | f_\lambda \rangle \right| \leqslant \mathrm{e}^{-at} \sum_{\mathrm{Re}\,\lambda < -a} | \langle f_{-\bar{\lambda}} | \varphi \rangle \langle \xi | f_\lambda \rangle | < \mathrm{e}^{-at} C
$$

for some constant $C$ and hence decay faster than $\mathrm{e}^{-at}$.

For long past times we work analogously from (5) rather than (4) to obtain

$$
\langle \xi | \hat{U}_t | \varphi \rangle \approx \langle \varphi | 1 \rangle \langle 1 | \xi \rangle + \sum_{0 < \mathrm{Re}\,(-\bar{\lambda}) \leqslant b} \mathrm{e}^{-\bar{\lambda} t} \langle f_\lambda | \varphi \rangle \langle \xi | f_{-\bar{\lambda}} \rangle \qquad \text{for} \quad t \to -\infty.
$$

Here the terms which decay faster than $\mathrm{e}^{bt}$ as $t \to -\infty$ have been discarded.

## 3. Surfaces of constant negative curvature

We will use the resolvent method described in the previous section to obtain a spectral decomposition for the evolution operator of a 'model' chaotic system. The simplest systems which are strongly chaotic (and in particular mixing) are the free motion on compact 2D surfaces of constant negative curvature.

The hyperbolic plane $N$ plays the role of a universal cover for these surfaces. It can be embedded in Minkowski space where the metric is

$$
\mathrm{d}s^2 = -\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2
$$

as the surface satisfying the equation

$$
-x_1^2 + x_2^2 + x_3^2 = -1.
$$

We shall consider the free motion on compactifications of $N$ formed by quotienting it under the action of some discrete group as described below. We start by analysing the free motion of a particle on the hyperbolic plane itself. This dynamical system has the phase space $T^*N$ and is described by the Hamiltonian $H = \frac{p^2}{2m}$, which is just the kinetic energy of the particle. The trajectories for this dynamical system are given by the geodesics of the surface $N$. For a particle with unit mass $m = 1$, the constant-energy surface $M$ with energy $E = \frac{1}{2}$ consists of the points with momenta of unit modulus (and hence also unit speed). Thus the energy shell $M$ is the unit-cotangent bundle of the hyperbolic plane. Changing the energy amounts to a rescaling which leaves the trajectories unchanged and only alters the rate at which they are traversed.

The group of isometries of the hyperbolic plane $G = SO(2, 1)$ acts simply transitively on the points of the energy shell. Let us take as a base point in $M$ the point $O = (q, p)$ where the position $q$ is given by $(0, 0, 1)$ and the momentum $p$ is $(1, 0, 0)$. A point $x$ on the energy shell may be identified with the unique element of $G$ which takes $O$ to $x = gO$. In this way the constant-energy shell can be identified with the elements of $G$. The above construction gives a diffeomorphism from the topological group $G$ to the energy shell $M$.

We choose a basis for the Lie algebra $g$ of $G$ consisting of the matrices

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (17)$$

An element $g \in G$ may be written as a product [14]

$$g(\phi, \tau, \psi) = e^{K\phi} e^{P\tau} e^{K\psi}. \quad (18)$$

The parameters $\tau$, $\phi$ and $\psi$ are known as the Euler angles. Note that $\tau$ and $\phi$ give the position in polar coordinates:

$$(x_1, x_2, x_3) = (\cosh \tau, \sinh \tau \cos \phi, \sinh \tau \sin \phi)$$

while the angle $\psi$ gives a consistent way of parametrizing the direction of the momentum.

After a time $t$ a free particle at the point $O$ on the constant energy surface advances along the geodesic to which it belongs to the point $h_t O$ where $h_t = e^{Pt}$. The point $gO$ moves to $gh_t O$; hence on identifying $M$ with $G$ the evolution corresponds to right multiplication by the element $h_t$. This one-parameter group of transformations

$$g \mapsto U_t(g) = gh_t$$

is the geodesic flow on $G$. The measure $\mu$ on $M$ which is invariant under the geodesic flow $U_t$ is the Haar measure $dg$ of the group $G$. This measure is invariant under right and left multiplication by elements of the group. The $U_t$ invariant scalar product for functions on $G$ is therefore given by

$$\langle f_1 | f_2 \rangle = \int_{g \in G} f_1(g) \overline{f_2(g)} \, dg$$
$$= \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} f_1(\phi, \tau, \psi) \overline{f_2(\phi, \tau, \psi)} \sinh \tau \, d\tau \, d\phi \, d\psi.$$

Finally we notice that right multiplication by elements in the rotation subgroup $H = \{e^{K\psi}\}$ only affects the direction of momentum and does not change the position of the particle. Therefore we can identify the hyperbolic plane $N$ with the set of right cosets $G/H$.

We now turn to the free motion on general surfaces of constant negative curvature which are constructed by taking a tessellation of the hyperbolic plane $G/H$ and identifying the tessellating shapes (the fundamental domains). Let $\Gamma$ be the group of transformations mapping the tessellating shapes to each other. For every $\gamma \in \Gamma$ the points $\gamma gH$ and $gH$ of the hyperbolic plane are identified. The points of the quotient surface formed under this identification are labelled by double cosets in $\Gamma \backslash G/H$. When the directions of momentum at each of the points are included the constant-energy surface $M = \Gamma \backslash G$ is formed. Since the free motion is given by the right shift by $h_t$ it is not affected by quotienting on the left by the subgroup $\Gamma$, so on the constant-energy surface $\Gamma \backslash G$ the geodesic flow is given by

$$U_t(\Gamma g) = \Gamma gh_t. \quad (19)$$

## 4. Decomposition of the evolution operator for the geodesic flows

In this section we use the representation theory of $G = SO(2, 1)$ to find a decomposition of the form (7) for the resolvent for the geodesic flow on $\Gamma\backslash G$.

The right regular representation $T_R(h)$ of $G$ on $L_2(\Gamma\backslash G)$ is defined by

$$T_R(h)\varphi(\Gamma g) = \varphi(\Gamma gh) \qquad \text{where} \quad \varphi \in L_2(\Gamma\backslash G).$$

It can be decomposed as a direct sum of irreducible unitary representations $T^y$, which leads to a splitting of $L_2(\Gamma\backslash G)$ into a direct sum of the spaces $\mathbb{H}(T^y)$ on which $T^y$ acts:

$$L_2(\Gamma\backslash G) = \bigoplus_{y\in Y} \mathbb{H}(T^y). \tag{20}$$

As the geodesic flow $U_t$ defined in (19) amounts to right multiplication by the group element $h_t$ the evolution operator $\hat{U}_t$ coincides with $T_R(h_t)$ and leaves $\mathbb{H}(T^y)$ invariant, leading to the decomposition

$$\hat{U}_t = \sum_{y\in Y} T^y(h_t). \tag{21}$$

Substituting (21) into (6) we obtain the resolvent for the geodesic flow:

$$R_-(z) = -\int_0^\infty e^{-zt}\hat{U}_t\,dt = \sum_{y\in Y} R_-^y(z) \tag{22}$$

where $R_-^y(z) = -\int_0^\infty e^{-zt}T^y(h_t)\,dt$.

The rest of this section is organized as follows. In section 4.1 we discuss the unitary irreducible representations $T^y$. In section 4.2 we study the decomposition (20) and relate it to the spectra of the Laplacian on the quotient surface $\Gamma\backslash G/H$. In section 4.3 we calculate the integrals $R_-^y(z)$ for each of these irreducible representations and in section 4.4 we combine all the results to find $R_-(z)$.

### 4.1. Irreducible representations of $SO(2,1)$

Let $T$ be an arbitrary unitary representation of $G = SO(2, 1)$ on the Hilbert space $\mathbb{H}(T)$ with scalar product $\langle\,|\,\rangle$. The Casimir operator is defined on $\mathbb{H}(T)$ by

$$\Omega(T) = L_P^2(T) + L_Q^2(T) - L_K^2(T)$$

where $L_X(T)$ is the Lie derivative of $T$ in the direction $X \in g$ and $P, Q, K \in g$ are as defined in (17). This operator commutes with each of the $T(g)$ [15] and therefore must be a scalar multiple of the identity on each of the irreducible representations $T^y$. It is convenient to denote by $T^\rho$ the unitary irreducible representation on which this scalar is $-\frac{1}{4} - \rho^2$, i.e. for all $v \in \mathbb{H}(T^\rho)$ we have

$$\Omega(T^\rho)v = (-\tfrac{1}{4} - \rho^2)v. \tag{23}$$

The following values of $\rho$ are allowed [15]:

- Im $\rho = 0$ and Re $\rho \geqslant 0$ (the principal series)
- Re $\rho = 0$ and Im $\rho \in (0, \frac{1}{2})$ (the complementary series)
- Re $\rho = 0$ and Im $\rho \in \mathbb{N}$ each corresponding to a pair of inequivalent representations (the discrete series).

In addition we have the one-dimensional identity representation $I$ for which $\Omega(I)v = 0$ for $v \in \mathbb{H}(I)$.

Under the action of the compact Abelian subgroup $H = \{e^{Kt}\}$ the representation $T^\rho$ splits into one-dimensional irreducible representations. Let $|n\rangle \in \mathbb{H}(T^\rho)$ be a vector in one such one-dimensional representation with

$$T^\rho(e^{Kt})|n\rangle = e^{int}|n\rangle. \tag{24}$$

It is a fact [16] that in a representation $T^\rho$ there is at most one vector $|n\rangle$ for each value $n$ and they can be normalized to form an orthonormal basis for the space $\mathbb{H}(T^\rho)$. We also use the notation $|\rho, n\rangle$, when the irreducible representation to which $|n\rangle$ belongs needs to be specified. For the representations of the principal and complementary series the value of $n$ ranges over all the integers. For the discrete series, one of the pair has a basis consisting of $|n\rangle$ where $n \geqslant \text{Im}\,\rho$ and the other where $n \leqslant \text{Im}\,\rho$ [16].

A vector $|\varphi\rangle$ in $H(T^\rho)$ may be expanded in the basis $\{|n\rangle\}$ as

$$|\varphi\rangle = \sum_n \langle n|\varphi\rangle|n\rangle. \tag{25}$$

In the basis $\{|n\rangle\}$ the matrix elements for the representation $T^\rho$ are given by [14]

$$\langle m|T^\rho(g)|n\rangle = e^{im\phi}e^{in\psi}B^l_{mn}(\cosh\tau)$$

where $l = \frac{1}{2} + i\rho$ and $B^l_{mn}(\cosh\tau)$ are the Jacobi functions.

### 4.2. Decomposition of $L_2(\Gamma\backslash G)$

We now consider the decomposition of $L_2(\Gamma\backslash G)$ into irreducible representations $T^\rho$ for a particular quotient surface $\Gamma\backslash G/H$.

Suppose that the decomposition (20) of $L_2(\Gamma\backslash G)$ has $N_\rho$ copies of the irreducible representation $T^\rho$, labelled by $s \in \{1, \ldots, N_\rho\}$:

$$L_2(\Gamma\backslash G) = \bigoplus_\rho \left( \bigoplus_{s=1}^{N_\rho} \mathbb{H}(T^{\rho;s}) \right). \tag{26}$$

Each $\mathbb{H}(T^{\rho;s})$ has a basis $\{|\rho, n; s\rangle\}$ as described in section (4.1). From (23) and (24) we deduce that the space spanned by the vectors $\{|\rho, n; s\rangle$ where $s = 1, \ldots, N_\rho\}$ is the intersection of the eigenspaces of $\Omega(T_R)$ and $L_K(T_R)$ with the eigenvalues $-\frac{1}{4} - \rho^2$ and $in$ respectively.

$$\Omega(T_R)|\rho, n; s\rangle = (-\tfrac{1}{4} - \rho^2)|\rho, n; s\rangle$$
$$L_k(T_R)|\rho, n; s\rangle = in|\rho, n; s\rangle. \tag{27}$$

The functions $|\rho, 0; s\rangle$ are invariant under multiplication on the right by elements of $H$ and therefore can be viewed as functions in $L_2(\Gamma\backslash G/H)$. On this space the Casimir operator $\Omega$ reduces to the Laplacian $\hat\Delta$, hence the values of $\rho$ occuring in (26) correspond to the eigenvalues $\epsilon(\rho) = -\frac{1}{4} - \rho^2$ of the Laplacian $\hat\Delta$ on the quotient surface $\Gamma\backslash G/H$. $N_\rho$ is the multiplicity of the eigenvalue $\epsilon(\rho)$ and $|\rho, 0; s\rangle$ are its eigenfunctions. It is known that the spectrum $\Lambda_\Gamma$ of $\hat\Delta$ on a compact surface $\Gamma\backslash G/H$ is discrete and is bounded above by zero. It also contains the eigenvalue 0, which corresponds to the the constant eigenfunction. Hence (26) takes the form

$$L_2(\Gamma\backslash G) = \bigoplus_{-\frac{1}{4} - \rho^2 \in \Lambda_\Gamma} \left( \bigoplus_{s=1}^{N_\rho} \mathbb{H}(T^{\rho;s}) \right). \tag{28}$$

A differential equation for the eigenfunctions $|\rho, n; s\rangle$ is obtained by unwrapping a function in $L_2(\Gamma\backslash G)$ to give a periodic function on the whole group $\chi : G \to \mathbb{C}$ which obeys

$$\chi(\gamma g) = \chi(g) \qquad \text{for all} \quad \gamma \in \Gamma.$$

Using Euler's coordinates (18) on $G$ in (27) we deduce that the unwrapping of the function $|\rho, n; s\rangle$ has the form $\chi_s^{\rho,n}(\tau, \theta)e^{in\psi}$, where $\chi_s^{\rho,n}(\tau, \theta)$ obeys the second-order differential equation

$$\left(\frac{1}{\sinh \tau} \frac{\partial}{\partial \tau} \sinh \tau \frac{\partial}{\partial \tau} + \frac{1}{\sinh^2 \tau}\left(\frac{\partial^2}{\partial^2 \phi} - 2in \cosh \tau \frac{\partial}{\partial \phi} - n^2\right)\right)\chi_s^{\rho,n} = \left(-\frac{1}{4} - \rho^2\right)\chi_s^{\rho,n}.$$

(29)

### 4.3. Spectral decompositions for the irreducible representations

We now find the correlation functions of the resolvents $R_-^\rho(z)$ for each of the irreducible representations $T^\rho$ and obtain a spectral decomposition for the operator $T^\rho(h_t)$.

Expanding

$$\langle \xi | R_-^\rho(z) | \varphi \rangle = -\int_0^\infty e^{-zt} \langle \xi | T^\rho(h_t) | \varphi \rangle \, dt$$

in the basis $\{|n\rangle\}$, we get

$$\langle \xi | R_-^\rho(z) | \varphi \rangle = \int_0^\infty e^{-zt} \sum_{m,n} \langle \xi | m \rangle \langle m | T^\rho(h_t) | n \rangle \langle n | \varphi \rangle \, dt.$$

(30)

The matrix element $\langle m | T^\rho(h_t) | n \rangle$ can be written as a sum over the exponentials $e^{\lambda t}$ where

$$\lambda \in \{l, l-1, l-2, \ldots; \bar{l}, \bar{l}-1, \bar{l}-2, \ldots\} = \Lambda_\rho$$

(31)

and

$$\langle m | T^\rho(h_t) | n \rangle = B_{mn}^l(\cosh t) = \sum_{\lambda \in \Lambda_\rho} c_{mn}^\lambda e^{\lambda t}.$$

(32)

Substituting (32) into the resolvent (30) we obtain

$$\begin{aligned}
\langle \xi | R_-^\rho(z) | \varphi \rangle &= \sum_{\lambda \in \Lambda_\rho} \int_0^\infty e^{(\lambda - z)t} \sum_{m,n} \langle \xi | m \rangle c_{mn}^\lambda \langle n | \varphi \rangle \, dt \\
&= \sum_{\lambda \in \Lambda_\rho} \sum_{m,n} \frac{\langle \xi | m \rangle c_{mn}^\lambda \langle n | \varphi \rangle}{\lambda - z} \\
&= \sum_{\lambda \in \Lambda_\rho} \frac{\langle \xi | \hat{K}_\lambda | \varphi \rangle}{\lambda - z}.
\end{aligned}$$

(33)

The operators $\hat{K}_\lambda$ determining the residue at the pole $\lambda \in \Lambda_\rho$ are given by

$$\hat{K}_\lambda = \sum_{m,n} |m\rangle c_{mn}^\lambda \langle n|.$$

The matrix elements $c_{mn}^\lambda$ may be split into a product (appendix B) $c_{mn}^\lambda = a_m^\lambda \overline{b_n^\lambda}$, which enables us to write (33) in the form

$$\begin{aligned}
\langle \xi | R_-^\rho(z) | \varphi \rangle &= \sum_{\lambda \in \Lambda_\rho} \frac{\sum_n \langle n | \overline{b_n^\lambda} | \varphi \rangle \sum_m \langle \xi | a_m^\lambda | m \rangle}{\lambda - z} \\
&= \sum_{\lambda \in \Lambda_\rho} \frac{\langle f_{-\bar{\lambda}} | \varphi \rangle \langle \xi | f_\lambda \rangle}{\lambda - z}
\end{aligned}$$

(34)

where $\lambda$ runs through the set $\Lambda_\rho$ (31) and

$$|f_\lambda\rangle = \sum_m a_m^\lambda |m\rangle \qquad |f_{-\bar{\lambda}}\rangle = \sum_m b_m^\lambda |m\rangle$$

are eigenvectors of $T^\rho(h_t)$:

$$T^\rho(h_t)|f_\eta\rangle = e^{\eta t}|f_\eta\rangle.$$

The coefficients $a_m^\lambda$ and $b_m^\lambda$ where $\lambda = l, \bar{l}$ are found explicitly in appendix B along with a procedure for generating the eigenfunctionals for the other values of $\lambda \in \Lambda_\rho$.

The eigenfunctionals $|f_\lambda\rangle$ and $|f_{-\bar{\lambda}}\rangle$ are linear functionals on the space $\mathbb{S}^\rho \subset \mathbb{H}(T^\rho)$ of test vectors $|\varphi\rangle = \sum_n c_n|\rho, n\rangle \in \mathbb{H}(T^\rho)$, where the coefficients $c_n \to 0$ as $|n| \to \infty$ faster than any power of $n$, i.e. $\lim_{|n|\to\infty} c_n n^k = 0$ for all $k \in \mathbb{N}$. By comparing (34) with (7) we arrive at the spectral decomposition for $T^\rho(h_t)$

$$T^\rho(h_t) = \sum_{\lambda \in \Lambda_\rho} e^{\lambda t}|f_\lambda\rangle\langle f_{-\bar{\lambda}}|. \tag{35}$$

This decomposition converges absolutely for $t > 0$ for test vectors in the dense subspace $\mathbb{T}^\rho \subset \mathbb{S}^\rho$ of vectors of the form $|\varphi\rangle = \sum_{n=-K}^{K} c_n|\rho, n\rangle \in \mathbb{H}(T^\rho)$ for some $K$ (appendix C).

### 4.4. Spectral decompositions of the evolution operators for the geodesic flows

On combining (35) with (28) we obtain the central result of this paper, a spectral decomposition for $\hat{U}_t$:

$$\hat{U}_t = T_R(h_t) = \sum_{-\frac{1}{4}-\rho^2 \in \Lambda_\Gamma} \sum_{\lambda \in \Lambda_\rho} \sum_{s=1}^{N_\rho} e^{\lambda t}|f_\lambda^{\rho;s}\rangle\langle f_{-\bar{\lambda}}^{\rho;s}| \tag{36}$$

where $\Lambda_\Gamma$ is the spectrum of the Laplacian on $\Gamma\backslash G/H$ containing eigenvalues $-\frac{1}{4}-\rho^2$ with multiplicities $N_\rho$ and where $\Lambda_\rho$ is given by equation (31) and also shown in figure 1. The functionals

$$|f_\lambda^{\rho;s}\rangle = \sum_n a_n^\lambda|\rho, n; s\rangle$$

$$|f_{-\bar{\lambda}}^{\rho;s}\rangle = \sum_n b_n^\lambda|\rho, n; s\rangle$$

are the eigenfunctionals of $\hat{U}_t$:

$$\hat{U}_t|f_\eta^{\rho;s}\rangle = e^{\eta t}|f_\eta^{\rho;s}\rangle.$$

The functions $|\rho, n; s\rangle$, $s = 1, \ldots, N_\rho$ form an orthogonal basis in the space of solutions of the linear differential equation (29). The coefficients are obtainable by expanding $B_{mn}^l(\cosh t)$ in powers of $e^{\lambda t}$ (see equation 32) and are given explicitly for $\lambda = l, \bar{l}$ in (B.7).

The eigenfunctionals $|f_\eta^{\rho;s}\rangle$ belong to the space $C^\infty(\Gamma\backslash G)^*$ and act on test functions in $\oplus_{-\frac{1}{4}-\rho^2 \in \Lambda_\Gamma} \mathbb{S}^\rho = C^\infty(\Gamma\backslash G)$. The decomposition (36) converges for functions in a dense subspace

$$\mathbb{T} = \left\{|\varphi\rangle = \sum_{-\frac{1}{4}-\rho^2 \in \Lambda_\Gamma} \sum_{s=1}^{N_\rho} \sum_{n=-K}^{K} c_{n,\rho;s}|\rho, n; s\rangle \text{ for some } K\right\} \subset C^\infty(\Gamma\backslash G) \tag{37}$$

of test functions whose unwrappings have only a finite number of Fourier components in the angle $\psi$ (appendix C).

The eigenvalues entering (36) could also be obtained by comparing classical orbit expansions for the traces of the resolvents of $\hat{L}$ and $\hat{\Delta}$ [17].
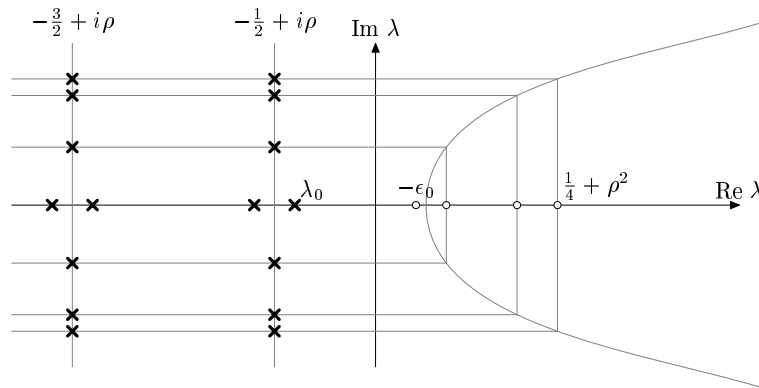
**Figure 1.** The poles $\lambda = -\frac{1}{2} - i\rho - k$ (crosses) of the resolvent give the eigenvalues $e^{\lambda t}$ of the evolution operator appearing in decomposition (36). These poles are related to the eigenvalues $-\epsilon = \frac{1}{4} + \rho^2$ of $-\hat{\Delta}$ (open circles). The rate of decay $e^{\lambda_0 t}$ at long times is determined by the pole $\lambda_0$ with the least negative real part.

## 5. Properties of the evolution operators for the geodesic flows

### 5.1. Decay of correlations

At long times only the leading terms with $k = 0$ in each of the $\Lambda_\rho$ in the spectral decomposition for $\hat{U}_t$ (36) remain significant (see section 2.4). Therefore as $t \rightarrow \infty$, for $\xi, \varphi \in \mathbb{T} \subset C^\infty(\Gamma \backslash G)$

$$\langle \xi | \hat{U}_t | \varphi \rangle \approx \sum_{-\frac{1}{4}-\rho^2 \in \Lambda_\Gamma} \sum_s e^{(-\frac{1}{2} \pm i\rho)t} \langle f_{\frac{1}{2} \pm i\rho}^{\rho;s} | \varphi \rangle \langle \xi | f_{-\frac{1}{2} \pm i\rho}^{\rho;s} \rangle.$$

Separating out the eigenvalue with $\lambda = 0$ for a compact quotient surface where the dynamics is mixing we find that

$$\hat{U}_t | \varphi \rangle \approx \langle 1 | \varphi \rangle \langle \xi | 1 \rangle + \sum_{0 \neq -\frac{1}{4}-\rho^2 \in \Lambda_\Gamma} \sum_s e^{(-\frac{1}{2} \pm i\rho)t} \langle f_{\frac{1}{2} \pm i\rho}^{\rho;s} | \varphi \rangle \langle \xi | f_{-\frac{1}{2} \pm i\rho}^{\rho;s} \rangle. \tag{38}$$

The approach to equilibrium is governed by the eigenvalue(s) $e^{(-\frac{1}{2} \pm i\rho)t}$ in this expansion with the largest modulus (see figure 1). There are two possibile cases distinguished by the smallest non-zero eigenvalue $-\epsilon_0$ of $-\hat{\Delta}$.

(1) If $-\epsilon_0 < \frac{1}{4}$ then the slowest-decaying term gives the decay rate $e^{\lambda_0 t}$, where $\lambda_0 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \epsilon_0}$. For the non-constant part of the correlation function $\langle \xi | \hat{U}_t | \varphi \rangle$ between $\xi, \varphi \in C^\infty(\Gamma \backslash G)$ we have

$$\langle \xi | \hat{U}_t | \varphi \rangle - \langle \xi | 1 \rangle \langle 1 | \varphi \rangle = O(e^{\lambda_0 t}). \tag{39}$$

(2) If $-\epsilon_0 > \frac{1}{4}$ then all the values of $\rho$ in (38) are real and we get $\lambda_0 = -\frac{1}{2}$

$$\langle \xi | \hat{U}_t | \varphi \rangle - \langle \xi | 1 \rangle \langle 1 | \varphi \rangle = O(e^{-\frac{t}{2}}). \tag{40}$$

By applying the result (14) to the geodesic flows we find that for arbitrary $\xi, \varphi \in C^\infty(M)$

$$|\langle \xi | \hat{U}_t | \varphi \rangle - \langle \xi | 1 \rangle \langle 1 | \varphi \rangle| < C e^{\lambda_0 t}.$$

This is a classical result, proving the exponential decay of correlations for geodesic flows on constant-negative-curvature surfaces (see e.g. [15] for a detailed discussion). Also from (14) we get the other terms in the long-time asymptote:

$$\left| \langle \xi | \hat{U}_t | \varphi \rangle - \langle \xi | 1 \rangle \langle 1 | \varphi \rangle - \sum_{-\frac{1}{4}-\rho^2 \in \Lambda_\Gamma} \sum_{s=1}^{N_\rho} \sum_{\pm} \sum_{k=0}^{N} e^{(-\frac{1}{2} \pm i\rho - k)t} \langle \xi | f_{-\frac{1}{2} \pm i\rho - k}^{\rho;s} \rangle \langle f_{\frac{1}{2} \pm i\rho + k}^{\rho;s} | \varphi \rangle \right|$$

$$< C(N) e^{(\lambda_0 - (N+1))t}. \tag{41}$$

### 5.2. Relation to the Laplacian

In addition to refining the long-time asymptotic form of the correlation function (41) the full expansion (36) provides a more detailed description of the evolution of distribution functions. Consider a smooth function $\Psi$ on $\Gamma \backslash G$ which is independent of the momentum coordinate $\psi$ (hence belongs to the space $\mathbb{T}$). As it evolves under $\hat{U}_t$ this function will acquire some angular dependence but we will only be interested in the density on the configuration space, i.e. the projection $\Psi_t = \hat{P} \hat{U}_t \Psi$ where $\hat{P}$ is the projector $\Gamma \backslash G \rightarrow \Gamma \backslash G / H$:

$$\hat{P} \chi = \frac{1}{2\pi} \int_0^{2\pi} \chi \, d\psi.$$

Expanding $\Psi$ as a linear combination of the eigenfunctions $|\rho, 0; s\rangle$ of the Laplacian for the quotient surface

$$|\Psi\rangle = \sum_{-\frac{1}{4}-\rho^2 \in \Lambda_\Gamma} \sum_s \langle \rho, 0, s | \Psi \rangle | \rho, 0; s \rangle$$

and using

$$\begin{aligned}
\hat{U}_t | \rho, 0; s \rangle &= T^\rho(h_t) | \rho, 0; s \rangle \\
&= \sum_n | \rho, n; s \rangle \langle \rho, n; s | T^\rho(h_t) | \rho, 0; s \rangle \\
&= \sum_n B_{n0}^{-\frac{1}{2}+i\rho} (\cosh t) | \rho, n; s \rangle
\end{aligned}$$

combined with the following expressions for the matrix elements of the projection operator:

$$\hat{P} | \rho, n; s \rangle = 0 \qquad n \neq 0 \qquad \hat{P} | \rho, n; s \rangle = | \rho, 0; s \rangle$$

we obtain for $\Psi_t$

$$|\Psi_t\rangle = \hat{P} \hat{U}_t |\Psi\rangle = \sum_{-\frac{1}{4}-\rho^2 \in \Lambda_\Gamma} B_{00}^{-\frac{1}{2}+i\rho} (\cosh t) \sum_s \langle \rho, 0, s | \Psi \rangle | \rho, 0; s \rangle. \tag{42}$$

Using the symbolic notation

$$\sqrt{-\hat{\Delta} - \frac{1}{4}} | \rho, 0; s \rangle = \rho | \rho, 0; s \rangle$$

we rewrite (42) as

$$\hat{P} \hat{U}_t |\Psi\rangle = B_{00}^{-\frac{1}{2}+i\sqrt{-\hat{\Delta}-\frac{1}{4}}} (\cosh t) |\Psi\rangle \tag{43}$$

which relates the Laplacian $\hat{\Delta}$ on the surfaces of constant negative curvature to the classical dynamics in the configuration space.

## 6. Conclusions

### 6.1. Summary of results

Decompositions for the evolution operators $\hat{U}_t$ of a general mixing dynamical system on a manifold $M$ were constructed following Ruelle's prescription [8] using the analytic continuation of the resolvent (6):

$$F_{\xi,\varphi}(z) = -\int_0^\infty e^{-zT} \langle \xi | \hat{U}_T | \varphi \rangle \, dT.$$

It was proved that when conditions (8), (9) are met we have the following decompostions of $\hat{U}_t$:

$$\langle \xi | \hat{U}_t | \varphi \rangle = \sum_\lambda e^{\lambda t} \sum_k \langle f^k_{-\bar{\lambda}} | \varphi \rangle \langle \xi | f^k_\lambda \rangle \qquad \text{convergent for} \quad t > 0$$

$$\langle \xi | \hat{U}_t | \varphi \rangle = \sum_\lambda e^{-\bar{\lambda} t} \sum_k \langle f^k_\lambda | \varphi \rangle \langle \xi | f^k_{-\bar{\lambda}} \rangle \qquad \text{convergent for} \quad t < 0$$

where $|f^k_\eta\rangle$ are eigenfunctionals of $\hat{U}_t$

$$\hat{U}_t | f^k_\eta \rangle = e^{\eta t} | f^k_\eta \rangle$$

belonging to the space $C^\infty(M)^*$. They can be obtained from the residues of $F_{\xi,\varphi}(z)$ which are positioned at $\lambda$ and have the form

$$2\pi i \mathrm{Res}(\lambda, F_{\xi,\varphi}) = \sum_k \langle \xi | f^k_\lambda \rangle \langle f^k_{-\bar{\lambda}} | \varphi \rangle.$$

On applying this method to the evolution operator $\hat{U}_t$ for the free motion on a compact surface of constant negative curvature $\Gamma \backslash G / H$ we obtained the decomposition (36) which is convergent for $\xi$ and $\varphi$ in a dense subspace $\mathbb{T} \subset C^\infty(\Gamma \backslash G)$ of functions whose unwrappings onto functions of $G$ have only a finite number of Fourier harmonics in the Euler angle $\psi$ (see equations (18), (37)). From this decomposition we obtained a refinement for the rate of decay of correlations (41) for these systems.

We also found that the projection of the evolution operator $\hat{P}\hat{U}_t$ from $\Gamma \backslash G$ to $\Gamma \backslash G / H$ is related to the Laplacian $\hat{\Delta}$ on the surface $\Gamma \backslash G / H$ by

$$\hat{P}\hat{U}_t = B_{00}^{-\frac{1}{2} + i\sqrt{-\hat{\Delta} - \frac{1}{4}}}(\cosh t).$$

### 6.2. Consequences for the ballistic $\sigma$-model

Finally we discuss some consequences of the spectral decomposition (4) for the ballistic $\sigma$-model [4,5]. Without diving into the (still controversial) issue of the $\sigma$-model derivation and its region of validity we quote below the result for the effective action (see [4, 5, 18] for a detailed discussion):

$$S = \int d\mu \, \mathrm{tr} \, \Lambda W^{-1} \{\hat{L} + i(\omega + i0)\Lambda\} W \tag{44}$$

where $W(x)$ belongs to some (super-) group and $\Lambda$ is a particular matrix in this group obeying $\Lambda^2 = 1$. The detailed structure of the target space is somewhat involved and for the purpose of our discussion it suffices to consider a toy model with $W \in SU(2)$ and $\Lambda = \mathrm{diag}(1, -1)$. Then the target space is a two-dimensional sphere $S^2 = SU(2)/U(1)$ parametrized by the matrices $Q = W^{-1}\Lambda W$. The action (44) does not depend on the parametrization of $S^2$ in

terms of $W$. Indeed, introducing the two-form $Q_* V$ on the energy shell $M$ obtained by pulling back the invariant volume $V$ on $S^2$ by the function $Q(x) : M \to S^2$ we get for the action

$$S = \int_M (Q_* V) \wedge p \, dq + i(\omega + i0) \, \mathrm{tr}(\Lambda Q) \, d\mu$$

where $p \, dq$ is the antiderivative of the simplectic structure $dp \wedge dq$. If $Q_* V$ is exact (and it is certainly closed) the first term does not depend on the choice of the antiderivative but only on the simplectic structure $dp \wedge dq$.

It is universally believed that some sort of regularization should supplement the effective action (44). It was conjectured [5] that in the limit of a vanishing regulator the eigenvalues of the regularized Liouvillian operator approach the Ruelle resonances (which were refered to as 'eigenvalues of the Perron Frobenius operator' in [5]).

We would like to point out that this conjecture must be further clarified due to the existence of the two inequivalent sets of Ruelle resonances. One set of resonances is in the left-hand half of the complex plane, while the other is in the right. These two sets originate from the different branches of the resolvent given by the two integral representations (equations (6) and (15) respectively). As a result there exist two non-equivalent regularizations of the operator $\hat{L}$. Denoting by $L_{\mathrm{reg}}$ the regularization of $\hat{L}$ with eigenvalues close to the Ruelle resonances in the left-hand half plane and observing that the operator $-L_{\mathrm{reg}}^+$ has eigenvalues close to the Ruelle resonances in the right-hand half plane we suggest that the two regularizations of $L$ are $L_{\mathrm{reg}}$ and $-L_{\mathrm{reg}}^+$.

We suggest the following structure for the regularized operator in the target space:

$$\hat{L} + i(\omega + i0)\Lambda \longrightarrow \begin{pmatrix} L_{\mathrm{reg}} + i\omega & 0 \\ 0 & -L_{\mathrm{reg}}^+ - i\omega \end{pmatrix} \tag{45}$$

which ensures the convergence of the action (44). Expanding the action (44) near $Q = \Lambda$ as $W = 1 + \begin{pmatrix} 0 & w \\ -\bar{w} & 0 \end{pmatrix}$ we verify that the choice (45) ensures that the quadratic part of the action $\delta^2 S$ is non-positively defined:

$$\delta^2 S = -\int d\mu \, (\bar{w}, w) \begin{pmatrix} 0 & L_{\mathrm{reg}} \\ -L_{\mathrm{reg}}^+ & 0 \end{pmatrix} \begin{pmatrix} w \\ \bar{w} \end{pmatrix}.$$

Note, that a similar structure for the regularized action appeared in the model with diffusive scattering [19].

## Acknowledgments

## Appendix A. Mixing dynamical systems

A dynamical system $(M, U_t, \mu)$ is mixing if for any two sets $A, B \subset M$

$$\lim_{t \to \infty} \mu(U_t A \cap B) = \mu(A)\mu(B). \tag{A.1}$$

For a mixing system the only square integrable eigenfunctions of $\hat{U}_t$ are the constant functions which have the eigenvalue 1. The proof [20] of this result is reproduced below.

We write (A.1) in terms of the characteristic functions $\chi_A$ and $\chi_B$ of the sets $A$ and $B$ (The characteristic function $\chi_C$ of a set $C$ takes on the value $\chi_C(x) = 1$ if $x \in C$ and 0 otherwise).

$$\lim_{t \to \infty} \langle \hat{U}_t \chi_A | \chi_B \rangle = \langle \chi_A | 1 \rangle \langle 1 | \chi_B \rangle. \tag{A.2}$$

Since the space of linear combinations of characteristic functions is dense in the space of square integrable functions (A.2) must also be true for $f, g \in L_2(M)$

$$\lim_{t \to \infty} \langle \hat{U}_t f | g \rangle = \langle f | 1 \rangle \langle 1 | g \rangle. \tag{A.3}$$

Let $f$ be an eigenfunction of $\hat{U}_t$ with eigenvalue $\mathrm{e}^{\lambda t}$ and $g = 1$, then (A.3) becomes

$$\mathrm{e}^{\lambda t} \langle f | 1 \rangle = \langle f | 1 \rangle$$

therefore the eigenvalue must be 1.

## Appendix B. Coefficients for the eigenfunctionals

We now study the matrix elements $c_{mn}^{\lambda} = \langle m | \hat{K}_{\lambda} | n \rangle$ of the operator $\hat{K}_{\lambda}$ appearing in the residues of the resolvent (33)

$$\langle \xi | R_{-}^{\rho}(z) | \varphi \rangle = \sum_{\lambda \in \Lambda_{\rho}} \frac{\langle \xi | \hat{K}_{\lambda} | \varphi \rangle}{\lambda - z} = \sum_{\lambda \in \Lambda_{\rho}} \sum_{m,n} \frac{\langle \xi | m \rangle c_{mn}^{\lambda} \langle n | \varphi \rangle}{\lambda - z}.$$

Consider the following operators on $\mathbb{H}(T^{\rho})$:

$$L = L_P(T^{\rho})$$
$$B_- = L_K(T^{\rho}) + L_Q(T^{\rho})$$
$$B_+ = L_K(T^{\rho}) - L_Q(T^{\rho}).$$

Note that $T^{\rho}(h_t) = \mathrm{e}^{Lt}$ and we have the commutation relations

$$[L, B_+] = B_+$$
$$[L, B_-] = -B_- \tag{B.1}$$
$$[B_-, B_+] = 2L.$$

From the commutation relations we have that $B_+ L = (L + 1)B_+$, therefore $B_+ \mathrm{e}^{Lt} = \mathrm{e}^{(L+1)t} B_+$, i.e. $B_+ T^{\rho}(h_t) = \mathrm{e}^t T^{\rho}(h_t) B_+$. By considering $\langle \xi | B_+ R_-^{\rho} | \varphi \rangle$ and using this result we obtain

$$B_+ \hat{K}_{\lambda} = \hat{K}_{\lambda+1} B_+. \tag{B.2}$$

Hence $B_+$ sends the image of $\hat{K}_{\lambda}$ into the image of $\hat{K}_{\lambda+1}$. In particular $B_+ \hat{K}_l = 0 = B_+ \hat{K}_{\bar{l}} = 0$ since there are no eigenvalues $l + 1$ and $\bar{l} + 1$ in the set $\Lambda_{\rho}$.

Consider an eigenfunctional $|f\rangle$ in the image of $K_{\lambda}$ so that it satisfies $T^{\rho}(h_t) | f \rangle = \mathrm{e}^{\lambda t} | f \rangle$ and therefore

$$L | f \rangle = \lambda | f \rangle.$$

Suppose also that it is annihilated by $B_+$:

$$B_+ | f \rangle = 0$$

so that using (B.1) we get

$$\langle n | B_+ | f \rangle = 0$$
$$\langle n | B_+ B_- | f \rangle = -2 \langle n | L | f \rangle = -2\lambda \langle n | f \rangle$$

for all $n \in \mathbb{Z}$.

Representing $|f\rangle$ by the linear combination

$$|f\rangle = \sum_n \alpha_n | n \rangle \qquad \text{where} \quad \alpha_n = \langle n | f \rangle$$

and using the explicit expression for the matrix elements of $B_+$ and $B_-$ [14]

$$-2\mathrm{i}\langle n|B_+ = -(l+n)\langle n-1| - 2n\langle n| + (l-n)\langle n+1|$$
$$-2\mathrm{i}\langle n|B_- = (l+n)\langle n-1| - 2n\langle n| - (l-n)\langle n+1| \tag{B.3}$$

we get the system of linear equations

$$-(l+n)\alpha_{n-1} - 2n\alpha_n + (l-n)\alpha_{n+1} = 0$$
$$-(l+n-1)(l+n)\alpha_{n-2} - 2(l+n)\alpha_{n-1} + 2(l(l+1)+n^2)\alpha_n$$
$$-2(l-n)\alpha_{n+1} - (l-n-1)(l-n)\alpha_{n-2} = -4\lambda\alpha_n.$$

These equations have only the trivial solution $\alpha_n = 0$ unless $\lambda = l = -\frac{1}{2} + \mathrm{i}\rho$ or $\lambda = \bar{l} = -\frac{1}{2} - \mathrm{i}\rho$ in which case they have the unique non-trivial solutions

$$\alpha_n^l = \frac{1}{(l+n)!(l-n)!} \qquad \alpha_n^{\bar{l}} = (-1)^n \tag{B.4}$$

respectively. Hence the images of $\hat{K}_l$ and $\hat{K}_{\bar{l}}$ are one dimensional.

Since $B_+$ sends the image of $\hat{K}_\lambda$ into the image of $\hat{K}_{\lambda+1}$ and its restriction on the image of $\hat{K}_\lambda$ has zero kernel for $\lambda \neq l, \bar{l}$ we deduce that the images of $\hat{K}_{l-k}$ and $\hat{K}_{\bar{l}-k}$ for all $k \in \mathbb{N}$ are also one dimensional. The operators $\hat{K}_\lambda$ where $\lambda \in \Lambda_\rho$ may therefore be written as

$$\hat{K}_\lambda = |f_\lambda\rangle\langle f_{-\bar{\lambda}}| = \sum_{m,n} |m\rangle a_m^\lambda \overline{b_n^\lambda}\langle n| = \sum_{m,n} |m\rangle c_{mn}^\lambda \langle n|$$

and their matrix elements $c_{mn}^\lambda$ split into a product $c_{mn}^\lambda = a_m^\lambda \overline{b_n^\lambda}$ where

$$|f_\lambda\rangle = \sum_m a_m^\lambda |m\rangle \qquad |f_{-\bar{\lambda}}\rangle = \sum_n b_n^\lambda |n\rangle \tag{B.5}$$

as stated in the main text.

Combining (B.4) with (B.5) we see that $a_n^l = \alpha_n^l$ and $a_n^{\bar{l}} = \alpha_n^{\bar{l}}$. This can be confirmed from equation (32) and the expansion of $B_{m0}^l(\cosh t)$ at large $t$ [21]:

$$B_{m0}^l(\cosh t) = \left(\frac{2^l(l!)^2}{\sqrt{\pi}} \frac{1}{(l+m)!(l-m)!} e^{lt} + \frac{2^{\bar{l}}(l!)}{\sqrt{\pi}} (-1)^m e^{\bar{l}t}\right)(1 + \mathrm{O}(e^{-t})). \tag{B.6}$$

Using (B.6) and the symmetry relation [14]

$$B_{mn}^l(\cosh t) = \frac{(l+n)!(l-n)!}{(l+m)!(l-m)!} B_{nm}^l(\cosh t)$$

we can find the coefficients $b_n^l$ and $b_n^{\bar{l}}$:

$$b_n^l = \frac{2^l}{\sqrt{\pi}}(l!)^2 \qquad b_n^{\bar{l}} = \frac{2^{\bar{l}}}{\sqrt{\pi}}(l!)^2(l+n)!(l-n)!.$$

To summarize, the functionals $|f_\lambda\rangle$, $|f_{-\bar{\lambda}}\rangle$ involved in the decomposition of $\hat{K}_\lambda$ for $\lambda = l, \bar{l}$ have the following form:

$$|f_l\rangle = \sum_n \frac{1}{(l+n)!(l-n)!}|n\rangle$$
$$|f_{\bar{l}}\rangle = \sum_n (-1)^n|n\rangle$$
$$|f_{-\bar{l}}\rangle = \sum_n \frac{2^l}{\sqrt{\pi}}(l!)^2|n\rangle \tag{B.7}$$
$$|f_{-l}\rangle = \sum_n \frac{2^{\bar{l}}}{\sqrt{\pi}}(l!)^2(l+n)!(l-n)!|n\rangle.$$

Given the eigenfunctionals $|f_l\rangle$ and $|f_{-\bar{l}}\rangle$ the eigenfunctionals $|f_\lambda\rangle$ for the other values of $\lambda \in \Lambda_\rho$ may be obtained by successive applications of the operator $B_-$:

$$|f_{\lambda-k}\rangle = B_-^k |f_\lambda\rangle \tag{B.8}$$

due to the relation $B_- \hat{K}_\lambda = \hat{K}_{\lambda-1} B_-$, which is proved analogously to (B.2).

Similarly, the projectors can be obtained by applying the operator $B_+$ to $|f_{-\bar{l}}\rangle$ and $|f_{-l}\rangle$:

$$|f_{-\bar{\lambda}+k}\rangle = B_+^k |f_{-\bar{\lambda}}\rangle. \tag{B.9}$$

From (B.3) and the relations (B.8), (B.9) we find that

$$
\begin{aligned}
a_n^{l-k} &= \langle n|f_{l-k}\rangle = \langle n|B_-^k|f_l\rangle = \mathrm{O}(n^k) \\
a_n^{\bar{l}-k} &= \langle n|f_{\bar{l}-k}\rangle = \langle n|B_-^k|f_{\bar{l}}\rangle = \mathrm{O}(n^k) \\
b_n^{l-k} &= \langle n|f_{-\bar{l}+k}\rangle = \langle n|B_+^k|f_{-\bar{l}}\rangle = \mathrm{O}(n^k) \\
b_n^{\bar{l}-k} &= \langle n|f_{-l+k}\rangle = \langle n|B_+^k|f_{-l}\rangle = \mathrm{O}(n^k).
\end{aligned}
\tag{B.10}
$$

## Appendix C. Convergence of the spectral decompositions

We now present the necessary condition for the convergence of the spectral decomposition (35):

$$\langle \xi|T^\rho(h_t)|\varphi\rangle = \sum_{\lambda\in\Lambda_\rho} \mathrm{e}^{\lambda t} \langle f_{-\bar{\lambda}}|\varphi\rangle\langle\xi|f_\lambda\rangle. \tag{C.1}$$

In particular we show it is convergent when $\xi$ and $\varphi$ belong to the dense subspace $\mathbb{T}^\rho$ of $S^\rho \subset \mathbb{H}(T^\rho)$ where

$$\mathbb{T}^\rho = \left\{ |\varphi\rangle = \sum_{n=-K}^{K} c_n|\rho,n\rangle \text{ for some } K \right\}. \tag{C.2}$$

First we show that each term $\mathrm{e}^{\lambda t}\langle f_{-\bar{\lambda}}|\varphi\rangle\langle\xi|f_\lambda\rangle$ in (C.1) is defined for $\xi$ and $\varphi$ in the subspace $\mathbb{S}^\rho$:

$$\mathbb{S}^\rho = \left\{ |\varphi\rangle = \sum_n c_n|\rho,n\rangle \text{ where } \lim_{|n|\to\infty} c_n n^q = 0 \text{ for all } q \in \mathbb{N} \right\}. \tag{C.3}$$

Note that $\mathbb{S}^\rho$ is invariant under $T^\rho(h_t)$. Let $\xi$ and $\varphi$ be given by

$$|\varphi\rangle = \sum_n c_n|\rho,n\rangle \qquad |\xi\rangle = \sum_n d_n|\rho,n\rangle. \tag{C.4}$$

Using $\langle f_{-\bar{\lambda}}|m\rangle\langle n|f_\lambda\rangle = a_m^\lambda \overline{b_n^\lambda}$ we get

$$|\mathrm{e}^{\lambda t}\langle f_{-\bar{\lambda}}|\varphi\rangle\langle\xi|f_\lambda\rangle| < \sum_{m,n} |\mathrm{e}^{\lambda t} a_m^\lambda \overline{b_n^\lambda} c_m \overline{d_n}|. \tag{C.5}$$

The eigenvalue $\lambda$ is of the form $l-k$ or $\bar{l}-k$ so by (B.10) $a_m^\lambda = \mathrm{O}(m^k)$ and $b_n^\lambda = \mathrm{O}(n^k)$. Hence

$$|\mathrm{e}^{\lambda t}\langle f_{-\bar{\lambda}}|\varphi\rangle\langle\xi|f_\lambda\rangle| < C|\mathrm{e}^{\lambda t}| \sum_m |c_m m^k| \sum_n |\overline{d_n}n^k|$$

which converges due to (C.3). Therefore each term is defined for $\xi, \varphi \in \mathbb{S}^\rho$ and the eigenfunctionals $|f_\eta\rangle$ belong to the space $\mathbb{S}^{\rho*}$.

Now we show that the sum (C.1) converges for $\xi, \varphi \in \mathbb{T}^\rho$. Since the sums (C.4) for $\xi, \varphi$ have only a finite number of terms it suffices to prove convergence for $|\varphi\rangle = |n\rangle$ and $|\xi\rangle = |m\rangle$. In this case (C.1) reduces to (32), which converges absolutely for $t > 0$.

Now we turn to the convergence of the spectral decomposition of the evolution operator $\hat{U}_t$ (36)

$$\langle\xi|\hat{U}_t|\varphi\rangle = \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \sum_{s=1}^{N_\rho} \sum_{\lambda\in\Lambda_\rho} e^{\lambda t} \langle\xi|f_\lambda^{\rho;s}\rangle\langle f_{-\bar{\lambda}}^{\rho;s}|\varphi\rangle. \tag{C.6}$$

By the above considerations the terms in this series $e^{\lambda t}\langle\xi|f_\lambda^{\rho;s}\rangle\langle f_{-\bar{\lambda}}^{\rho;s}|\varphi\rangle$ are defined for $\xi, \varphi \in \oplus_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \mathbb{S}^\rho = C^\infty(\Gamma\backslash G)$.

We will show that expansion (C.6) is convergent when $\xi, \varphi$ belong to the subspace $\mathbb{T} \subset C^\infty(\Gamma\backslash G)$:

$$\mathbb{T} = \left\{ |\varphi\rangle = \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \sum_{s=1}^{N_\rho} \sum_{n=-K}^{K} c_{n,\rho;s}|\rho, n; s\rangle \text{ for some } K \right\}. \tag{C.7}$$

It is sufficient to determine the convergence for

$$|\varphi\rangle = \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \sum_{s=1}^{N_\rho} c_{\rho;s}|\rho, n; s\rangle$$

$$|\xi\rangle = \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \sum_{s=1}^{N_\rho} d_{\rho;s}|\rho, m; s\rangle$$

for which decomposition (C.6) is

$$\langle\xi|\hat{U}_t|\varphi\rangle = \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \sum_{s=1}^{N_\rho} \sum_{\lambda\in\Lambda_\rho} e^{\lambda t} c_{\rho,s} a_n^\lambda \overline{d_{\rho,s}} b_m^\lambda. \tag{C.8}$$

Let $\varphi(\tau, \theta, \psi) = \varphi_n(\tau, \theta)e^{in\psi}$ be the unwrapping of $|\varphi\rangle$. The function $\varphi_n$ on $\Gamma\backslash G/H$ has the norm

$$\|\varphi_n\|^2 = \frac{1}{2\pi} \int_F |\varphi_n(\tau, \theta)|^2 \sinh\tau \, d\tau \, d\theta \qquad F \text{ is a fundamental domain.}$$

We may expand $\varphi_n$ in the eigenfunctions $\chi_n^{\rho;s}$ (the unwrappings of $|\rho, n; s\rangle$) of the operator (29):

$$\varphi_n(\tau, \theta) = \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \sum_{s=1}^{N_\rho} c_{\rho,s} \chi_n^{\rho;s} \tag{C.9}$$

hence

$$\|\varphi_n\|^2 = \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \sum_{s=1}^{N_\rho} |c_{\rho;s}|^2.$$

Similarly

$$\|\xi_n\|^2 = \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \sum_{s=1}^{N_\rho} |d_{\rho;s}|^2.$$

Using the fact that

$$\sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma} \sum_{s=1}^{N_\rho} |c_{\rho,s}\overline{d_{\rho,s}}| < \|\xi_n\|\|\varphi_n\|$$

and that (32) converges absolutely we see from (C.8) that

$$|\langle\xi|\hat{U}_t|\varphi\rangle| = \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma}\sum_{s=1}^{N_\rho}|c_{\rho,s}\overline{d_{\rho,s}}|\sum_{\lambda\in\Lambda_\rho}|a_n^\lambda\overline{b_m^\lambda}e^{\lambda t}| < C \sum_{-\frac{1}{4}-\rho^2\in\Lambda_\Gamma}\sum_{s=1}^{N_\rho}|c_{\rho,s}\overline{d_{\rho,s}}| < C\|\xi_n\|\|\varphi_n\|$$

is bounded.

Hence taking $\xi,\varphi\in\mathbb{T}$ ensures the convergence of the spectral decomposition (C.6). Since the unwrapping of $|\rho,n;s\rangle$ has the form $\chi_n^{\rho;s}$ the unwrappings of functions in $\mathbb{T}$ will have only a finite number of Fourier components in $\psi$.

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